university of groningen

Exam Computer Assisted Problem Solving (CAPS)

April 5th 2017 14.00-17.00

This exam is 'closed book'. It is NOT allowed to use a book or lecture notes. Only the use of a simple pocket calculator is allowed. Programmable calculators are not allowed, nor the use of electronic devices (tablet, laptop, etc.).

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: [10]

1. The problem $\cos(x) = x^2$ has a solution $x \approx 0.82$.

   (a) [5] (1) Give the iteration formula when Newton's method is used for this problem.
   (2) Compute the first two iterations (x_1 and x_2) with Newton's method if $x_0 = 1$.

   (b) [3] Someone uses the iterative method $x_{n+1} = \sqrt{\cos(x_n)}$, with $x_0 = 1$.
   Will this be a fast method? Explain why (not).

   (c) [10] When the iterative method $x_{n+1} = x_n + \frac{1}{2}(\cos(x_n) - x_n^2)$ is used, again with $x_0 = 1$, the first 4 iterations are given by

n	x_n
0	1.0
1	0.77015115
2	0.83248747
3	0.82248882
4	0.82444459

   (1) What are the convergence order and factor?
   (2) Determine an error estimate for x_4.
   (3) Calculate an improved solution for x_4 by means of Steffensen extrapolation.
   (4) Does the parameter $\alpha = \frac{1}{2}$ give optimal linear convergence?
   If not, determine the optimal value for α.

   (d) [8] Give a complete program (in pseudo code or Matlab-like language) that solves the problem using the $\alpha = \frac{1}{2}$ method described in (c), with(!) Steffensen extrapolation, with an accuracy of tol=1E-6.
   Use an appropriate stopping criterion and start-up procedure.

2. To compute the value of ln(2) one could use a numerical method to compute $\int_1^2 \frac{1}{x}\,dx$

   (a) [6] Suppose a grid is used with 5 segments.
   (1) What is the sub-area for the middle segment if the Midpoint method is used?
   Determine the local error for that sub-area.
   (2) What is the sub-area for the middle segment if Simpson's rule is used?

   The Trapezoidal method yields the following results

n	$I(n)$
16	0.6933912
32	0.6932082
64	0.6931624
128	0.6931510

   $I(n)$ is the approximation of the integral on a grid with n sub-intervals.

P.T.O.

(b) 10 (1) Will the Trapezoidal method give optimal convergence for $n = 256, 512, ...$? Explain why.
(2) Compute the q-factor and give error estimates for $I(128)$ based on both subsequent $I(n)$ values and on the global error theorem.
(3) What will be the error approximately in case of $n = 1024$ segments?
(4) Compute improved solutions $T_2(128)$ and $T_3(128)$ by means of extrapolations.

(c) 9 Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using the Trapezoidal method (no extrapolation). Use an appropriate error estimate for the stopping criterion.
Your program should be as computationally efficient as possible.

3. Consider the differential equation $y'(x) = -4y + 3x + 2$, with boundary condition $y(0) = 1$.

(a) 7 Compute the solution at $x = 1.0$ on a grid with $\Delta x = 1.0$:
(1) with Heun's method (RK2)
(2) with the Trapezoidal (Crank-Nicolson) method

With the RK4 method the solution is determined on 2 grids with $\Delta x = 0.25$ and $\Delta x = 0.125$. The result at a selection of x locations is as follows

x_n	$\Delta x = 0.25$	$\Delta x = 0.125$
0.00	1.00000000	1.00000000
0.50	0.78417969	0.78069047
1.00	1.07609558	1.07513195
1.50	1.43941188	1.43921226
2.00	1.81276886	1.81273210

(b) 7 (1) Give an error estimate and an improved solution (extrapolation) for the solution at $x = 1.5$ on the fine grid.
(2) Which grid Δx (use halving of gridsize) is required for an error of 1.0E-8?
(3) Are the employed gridsizes $\Delta x = 0.25$ and 0.125 within the stability limits?

(c) 4 The Midpoint method for integration leads to the following ODE method

$$y_{n+1} - y_n = hf(\frac{x_n + x_{n+1}}{2}, \frac{y_n + y_{n+1}}{2})$$

(1) Derive the formula for the absolute stability region (general ah).
(2) Determine the maximum step size (if any) for the given differential equation.

(d) 8 Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using the Heun method (without extrapolation). Use an appropriate error estimate for the stopping criterion.

4. Consider the diff. eqn. $2x\,y''(x) + \sin(\pi\,x)\,y(x) = \sqrt{8x}$, with boundary conditions $y(0) = 1$ and $y(1) = 0$.

(a) 7 (1) Give the matrix and rhs-vector when the matrix method is used with the [1 -2 1]-formula for $y''(x)$ on a grid with N=2 segments (one interior point).
(2) Compute the solution in the interior point.

(b) 6 The [1 -2 1]-formula for $y''(x)$ can be derived by considering $y''(x) = \frac{d}{dx}y'(x)$ on an equidistantly spaced grid with gridpoints $x_{[i-1]}$, $x_{[i]}$ and $x_{[i+1]}$, or with Taylor series.
(1) Derive a formula (many options) for a stretched grid with $N = 2$ segments, with $x_{[i+1]} - x_{[i]} = 2\,(x_{[i]} - x_{[i-1]})$.
(2) Describe the changes in the matrix and rhs-vector when this grid is used ($N = 2$).

Total: 100